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Albeverio and Hoegh-Krohn approach to a p-adic functional integration*

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Abstract. The theory of p-adic valued Gaussian and Feynman integration on infinite-dimensional spaces is developed with the aid of the Albeverio and Hoegh-Krohn approach based on the Fourier (or Laplace) transform of the infinite-dimensional distributions. Gaussian distributions cannot be realized as bounded p-adic valued measures and there is no difference between Gaussian and Feynman integration in this case. The free p-adic valued quantum field is realized in the space of Gaussian square integrable functions.

1. Introduction

Much interest in p-adic physics has arisen in connection with string theory [1]. The pioneer article by Volovich [2] generated a series of papers on p-adic string theory (see, for example, [3–9]). However, there were problems with the physical interpretation of such a high level model as the p-adic string and hence simpler models such as p-adic quantum mechanics and field theory were investigated in [8–13].

The main ideology of p-adic physics is to create the foundations of this physics in a similar way to that for standard real physics and then to return step by step to the p-adic string model by using the apparatus of p-adic quantum mechanics and field theory.

There are two classes of p-adic physical models. The first type of model is based on complex valued wavefunctions with a p-adic argument, $\varphi : Q_p^n \to \mathbb{C}$, where Q_p is the field of p-adic numbers (p-adic variables and a complex valued wavefunction); wavefunctions $\varphi : Q_p^n \to Q_p(\sqrt{\kappa})$ (where $Q_p(\sqrt{\kappa})$ is one of the quadratic extensions of Q_p), are considered in the second type of model. The complex valued p-adic quantum mechanics and field theory were developed by Vladimirov and Volovich [10, 11] (see also [8]). The p-adic valued p-adic quantum mechanics was studied in [12, 13] and some results for p-adic valued quantum field theory were proposed in [14].

The role of Gaussian integrals on infinite-dimensional spaces in the real valued quantum field theory is well known (see, for example, [15–17]). We also begin our considerations from the theory of infinite-dimensional Gaussian integration in the p-adic case. In the present paper we study the representation of the p-adic valued quantum field in the space $L_2(S_{-\infty}, \mathrm{d}\nu)$ of square integrable functions with respect to the p-adic valued Gaussian distribution ν.

This distribution is introduced with the aid of the theory of infinite-dimensional non-Archimedean distributions [9]. This definition is very similar to the definition of the

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Feynman path integral proposed by Albeverio and Hoegh-Krohn [18]. We integrate with respect to the Gaussian distribution Laplace transformations of the p-adic valued infinite-dimensional distributions in a way similar to that used to integrate the Fourier transforms of the bounded measures on Hilbert spaces in [18] (some results from Feynman integration inside infinite-dimensional theories of distributions are contained in the papers by Smolaynov *et al* [19–23] and, on the physical level, the definition of a Feynman integral without limiting procedure was proposed by De Witt-Morette [24]).

Probably, it is a unique way to introduce the p-adic valued analogue of the Gaussian measure. It would be impossible to define this object as the bounded measure (this mathematical result was proved in [25]). We must also note that in the p-adic valued case there is no difference between Gaussian and Feynman integrals because the square root $\mathrm{i} = \sqrt{-1} \in Q_p$ for some prime numbers p (see, for example, [26, 27]). In the same reasoning there is no difference between $(-, +, +, +)$ and $(+, +, +, +)$ spacetimes. The p-adic extension of the approach [18] is probably a unique way of introducing a p-adic Feynman integral because there is no possibility of using, for example, a generalization of the original Feynman definition based on the limiting procedure, as there is also no Lebesgue measure on finite-dimensional p-adic spaces [26, 27].

2. Fields of p-adic numbers and their quadratic extensions

It is known (see, for example, [26, 27]), that any norm on Q is equivalent to the usual absolute value or to a p-adic norm. The p-adic norm is defined in the following way. Let p be a prime number, $p = 2, 3, 5, \ldots$. Any non-zero rational number x can be represented in the form $x = p^r m/n$,where m and n are integers that are not divisible by p. Then the p-adic norm is $|x|_p = p^{-r}$ and $|0|_p = 0$. This norm satisfies the inequality $|x + y|_p \leqslant \max(|x|_p, |y|_p)$, i.e. it is non-Archimedean [26, 27]. The completion of Q with respect to the p-adic norm defines the p-adic number field Q_p. Any p-adic number can be uniquely represented in the canonical series:

$$a = p^{-n}a_{-n} + \cdots + p^{-1}a_{-1} + a_0 + pa_1 + \cdots + p^k a_k + \cdots \tag{1}$$

where $a_j = 0, 1, \ldots, p-1$. This series converges in the p-adic norm because $|p^n|_p = p^{-n}$.

The p-adic exponential is defined by the series

$$\exp\{x\} = \sum_{n=0}^{\infty} x^n/n!.$$

It is known [26, 27] that the region of convergence of the exponential function is

$$\{x \in Q_p : |x|_p < p^{1/(1-p)}\}.$$

The main problem in the p-adic case is the increase in $1/|n!|_p$ when $n \to \infty$. In this case the estimate $1/|n!|_p \leqslant p^{n/(p-1)}$ holds and the exponential is not an entire analytic function in the p-adic case. It is analytic only on the ball with its centre at zero.

The basis of p-adic valued quantum models is a p-adic Hilbert space (which was introduce in [9, 12, 13]). Let us recall the definition of p-adic Hilbert space. At first we must consider a p-adic analogue of complex numbers. As we know the field of complex numbers $\mathbb{C}$ is the quadratic extension of $\mathbb{R}$. In this case we have a very simple algebraic structure because this quadratic extension is, at the same time, the algebraic closure of the field of real numbers. In the p-adic case such a simple structure does not exist and there is no unique quadratic extension as in the real case. For $p = 2$ there are seven different quadratic extensions and for $p \neq 2$ there are three different quadratic extensions. All these quadratic extensions are not algebraically closed and extensions of any finite order are not algebraically closed. The algebraic closure of Q_p is constructed as an infinite chain of extensions of finite orders. However, this algebraic closure is not a complete field so we must consider the completion of this field. This is the final step in this long procedure because this completion is an algebraically closed field. Let us denote this field by C_p. In the mathematical literature this field is called the field of complex p-adic numbers but we shall not use C_p as a field of complex p-adic numbers. As we know in usual quantum mechanics a major role is played by the automorphism $* : C_p \to C_p, z \to z^*$. The real number field is invariant under the action of this automorphism and so is the expression $|z|^2 = zz^*$ which is considered in quantum mechanics as a probability. This automorphism is connected with the charge of the quantum particle. The action of $*$ changes the sign of the charge of the particle and, in quantum field theory, this automorphism corresponds to the process: particle $\to$ antiparticle. As the probability is an invariant of $*$ there is a particle–antiparticle symmetry in quantum physics. That is why we need an analogue of $*$ in p-adic valued quantum mechanics. The simplest possibility in this direction is to use a quadratic extension of Q_p with an analogue of the automorphism of complex conjugation. Hence we will use quadratic extensions of Q_p as the basis of our construction. There are a number of different quadratic extensions and there will be a number of different representations of the p-adic valued quantum mechanics. The next step in this direction is to consider the group $\{1, *\}$ as the simplest Galois group and to generalize our construction to the case of an arbitrary Galois extension of Q_p (see [9, p 120]). However, it is currently impossible to use C_p for the same considerations because the description of a group of automorphisms of C_p is an unsolved mathematical problem. In addition this group is an infinite group and it would be impossible to create an invariant which would be a generalization of zz^*.

Let us consider a quadratic equation $x^2 - \kappa = 0, \kappa \in Q_p$, which has no solution in Q_p. Let us denote by Z_κ the quadratic extension $Q_p(\sqrt{\kappa}) : z = x + \sqrt{\kappa}y, x, y \in Q_p, z^* = x - \sqrt{\kappa}y, |z|^2 = zz^* = x^2 - \kappa y^2 \in Q_p$. The extension of the p-adic valuation to Z_κ is defined by $|z|_p = \sqrt{||z|^2|_p}$.

3. p-adic Hilbert space

Let us consider a sequence $\lambda = (\lambda_n), \lambda_n \in Q_p, \lambda_n \neq 0$. We denote by H_λ the space of sequences

$$\left\{ f = (f_n)_{n=1}^{\infty} : f_n \in Z_\kappa \text{ and } |f|_\lambda^2 = \sum_{n=1}^{\infty} |f_n|^2 \lambda_n \text{ is a converging series in } Q_p \right\}.$$

Let us define on the space H_λ a generalization of an inner product:

$$(f, g)_\lambda = \sum_{n=1}^{\infty} f_n g_n^* \lambda_n.$$

Let us set $|f|_\lambda^2 = (f, f)$. The basis vectors $e^j = (e_i^j) = (\delta_i^j)$ are orthogonal vectors with respect to this inner product.

Let us consider the non-Archimedean [27] norm defined by

$$\|f\|_\lambda = \max_{1 \leqslant n \leqslant \infty} |f_n|_p \sqrt{|\lambda|_p}.$$

The space H_λ with this norm is a non-Archimedean Banach space and

$$|(f, g)_\lambda|_p \leqslant \|f\|_\lambda \|g\|_\lambda.$$

We must note that the inner product in H_λ does not possess some of the usual properties of the standard complex inner product on usual Hilbert spaces. For example, there is no p-adic analogue of the inequality $(f, f)_\lambda \geqslant 0$. We must also note that $(f, f)_\lambda = |f|^2_\lambda = 0$ for some $f \in H_\lambda$, $f \neq 0$.

This is why we define an inner product on a linear space E over a field Z_κ as an arbitrary Hermitian bilinear form: $(f, g) = (g, f)^*$.

Let us consider a triple $(E, \|\cdot\|, (\cdot\,,\cdot))$ where E is a non-Archimedean Banach space with norm $\|\cdot\|$ and $(\cdot\,,\cdot)$ is an inner product on E. This triple is called [12, 13, 9] a Z_κ-Hilbert space if it is isomorphic to a canonical triple $(H_\lambda, \|\cdot\|_\lambda, (\cdot\,,\cdot)_\lambda)$ for some $\lambda \in Q_p^\infty$:

$$I : E \to H_\lambda$$
$$\|Ix\|_\lambda = \|x\|$$
$$(Ix, Ix)_\lambda = (x, y).$$

In the same way we define Q_p-Hilbert space using the spaces of Q_p-sequences (we use the same symbols to denote these spaces). In particular, we need in the Q_p-Hilbert space $l_2 = l_2(Q_p)$ of square summable p-adic sequences, $\lambda_n = 1$ for all n. p-adic Hilbert spaces which are not isomorphic to l_2 exist.

We also wish to discuss briefly the problem of the normalization of basis vectors e^j in H_λ. We have $|e^j|^2 = (e^j, e^j)\lambda_j$. If all weight coefficients λ_j are positive rational numbers, there is no problem in normalizing the basis vectors e^j in ordinary real (or complex) Hilbert space setting $e^j_{\mathrm{norm}} = e^j/\sqrt{\lambda_j}$. However, in the p-adic case it is also possible that $\sqrt{\lambda_j}$ does not exist in the case of rational positive λ_j. This problem will arise in the L_2-construction with respect to the Gaussian measure, where we cannot normalize the Hermitian polynomials $h_\alpha(\phi)$ and need to use weight sequence $\lambda_\alpha = |h_\alpha|^2 = \alpha!2^\alpha$.

4. Distributions on infinite-dimensional p-adic spaces

Let us denote by S_k, $k \in Z$, the p-adic Hilbert space $H_{(p^k)}$, so

$$S_k = \left\{ f = (f_1, \ldots, f_n, \ldots) \in Q_p^\infty : \text{ and } |f|^2_{(p^{-k})} = \sum_{n=1}^{\infty} f_n^2 p^{-kn} \text{ is a converging series in } Q_p \right\}$$

and, in particular, $S_0 = l_2$. Let us use the symbol $\|\cdot\|_k$ for the norm on S_k. It is evident that

$$\cdots \subset S_k \subset S_{k-1} \subset \cdots \subset S_0 \subset \cdots \subset S_{-k} \subset S_{-(k+1)} \subset \cdots$$

with continuous embedding. Thus, we have a p-adic analogue of the concept of a nested Hilbert space.

Now let us introduce the spaces of sequences $S_\infty = \bigcap_{k=o}^{\infty} S_k$ and $S_{-\infty} = \bigcup_{k=-\infty}^{0} S_k$. The space S_∞ will be endowed with the projective topology $S_\infty = \lim_{k\to\infty} \operatorname{proj} S_k$ and $S_{-\infty}$ is endowed with inductive one $S_{-\infty} = \lim_{k\to-\infty} \operatorname{ind} S_k$. We need the following topological proposition:

Theorem 4.1. The space S'_∞ dual to S_∞ is isomorphic to $S_{-\infty}$.

Now we are interested in analytic functions on infinite dimensional p-adic spaces.

Let E be a locally convex Z_κ-linear space and $\{\|\cdot\|_\alpha\}$ be a system of non-Archimedean seminorms which define the topology on E;

$$U_{\alpha,\rho} = \{x \in E : \|x\|_\alpha \leqslant \rho\}$$

is the system of balls with its centre at the zero of E. These balls are simultaneously open and closed ('clopen').

Remark 4.1. The seminorm $\|\cdot\|$ is called a non-Archimedean seminorm (see [27]) if the strong triangle inequality is valid for it:

$$\|x+y\| \leqslant \max\{\|x\|, \|y\|\}.$$

Definition 4.1. A function $F : U_{\alpha,\rho} \to Z_\kappa$ is called an analytic function on this ball if it can be expanded into the power series:

$$F(x) = \sum_{n=0}^{\infty} B_n(x, \ldots, x) \tag{2}$$

where $B_n : E \times \cdots \times E \to Z_\kappa$ are continuous symmetric n-linear forms, and

$$\lim_{n\to\infty} \sup_{x_j \in U_{\alpha,\rho}} |B_n(x_1, \ldots, x_n)|_p = 0. \tag{3}$$

According to (3) the series (2) converges uniformly on the ball $U_{\alpha,\rho}$.

Let us denote by the symbol $A\left(U_{\alpha,\rho}\right)$ the space of analytic functions on the ball $U_{\alpha,\rho}$. A function F is called analytic at the zero if $F \in A(U_{\alpha,\rho})$ for some α and ρ and F is called entire analytic if $F \in A(U_{\alpha,\rho})$ for every α and ρ. Let us denote by $A_0(E)$ the space of functions which are analytic at the zero and by $A(E)$ the space of entire analytic functions.The first space is endowed with the inductive topology

$$A_0(E) = \lim_{\alpha,\rho} \operatorname{ind} A(U_{\alpha,\rho})$$

and the second one with the projective topology

$$A(E) = \lim_{\alpha,\rho} \operatorname{proj} A(U_{\alpha,\rho}).$$

We will use the functional spaces $A_0(S_\infty)$ and $A(S_{-\infty})$ as test function spaces on the sequence spaces S_∞ and $S_{-\infty}$ respectively and the dual spaces $A_0'(S_\infty)$ and $A'(S_{-\infty})$ as spaces of distribution.

Definition 4.2. The Laplace transformation of the distribution $\mu \in A_0'(S_\infty)$ is defined by the equality:

$$L(\mu)(g) = (\mu, \exp\{(g, \cdot)\}) \qquad g \in S_{-\infty}.$$

There is no problem in proving that $\exp\{(g, \cdot)\} \in A_0(S_\infty)$ for all $g \in S_{-\infty}$.

Remark 4.2. As we noted in section 2 the p-adic exponential function is not an entire analytic function (it is defined only on a small ball with its centre at zero) and this is why we need in the theory of locally defined test functions.

Theorem 4.2. The Laplace transformation is an isomorphism of the space of distributions $A_0'(S_\infty)$ and the space of test functions $A(S_{-\infty})$.

As we have a continuous linear operator $L : A_0'(S_\infty) \to A(S_{-\infty})$ we can define the adjoint operator $L' : A'(S_{-\infty}) \to A_0(S_\infty)$.

We will define the Gaussian distribution on the infinite dimensional p-adic space as a distribution belonging to $A'(S_{-\infty})$. Let $b : S_\infty \times S_\infty \to Z_\kappa$ be a symmetric continuous bilinear form and $a \in S_\infty$.

Definition 4.3. The Gaussian distribution $\nu_{b,a}$ with the covariance b and the mean value a is defined as an element of $A'(S_{-\infty})$ with the L'-transformation:

$$L'(\nu_{b,a})(f) = \exp\{b(f, f)/2 + (f, a)\}.$$

Remark 4.3. There is a plus sign before the covariance form because we use the Laplace transformation.

We define a generalized $\nu_{b,a}$-integral with the aid of the equality (cf [18]):

$$\int_{S_{-\infty}} F(\phi)\nu_{b,a}(\mathrm{d}\phi) = (\nu_{b,a}, F).$$

5. The space of the square integrable functions with respect to the infinite-dimensional Gaussian distribution

Now we are interested in the Gaussian distribution $\nu : a = 0$, $b(f, f) = (f, f)/2 = (1/2)\sum_{n=1}^{\infty} f_n^2$ and we shall define the space $L_2(S_{-\infty}, \mathrm{d}\nu)$ of square integrable functions $F : S_{-\infty} \to Z_\kappa$ with respect to ν.

Let us introduce on the functional space $A(S_{-\infty})$ the inner product

$$(F, G) = \int_{S_{-\infty}} F(\phi)G^*(\phi)\,\mathrm{d}\nu\,(\phi). \tag{4}$$

This is a continuous Hermitian bilinear form on $A(S_{-\infty})$.

Let us denote by $\{h_n(t)\}_{n=o}^{\infty}$ the system of usual Hermitian polynomials. It well known that the coefficients of these polynomials belong to the field of rational numbers and there is no problem in considering these polynomials as functions of the p-adic argument, $t \in Q_p$. As usual we can consider the Hermitian polynomials $h_\alpha(\phi)$ on the infinite-dimensional space of sequences:

$$h_\alpha(\phi) = h_{\alpha_1}(\phi_1)\ldots h_{\alpha_n}(\phi_n)\ldots$$

where $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, $\alpha_j = 0, 1, 2, \ldots$ and $|\alpha| = \sum_{j=1}^{\infty} \alpha_j < \infty$.

Proposition 5.1. The Hermitian polynomials $\{h_\alpha\}$ are orthogonal with respect to the inner product (4): $\int_{S_{-\infty}} h_\alpha(\phi)h_\beta(\phi)\,\mathrm{d}\nu\,(\phi) = \delta_{\alpha\beta}2^{|\alpha|}\alpha!$

We cannot normalize the Hermitian polynomials in the p-adic case.

Theorem 5.1. The system of Hermitian polynomials is a topological basis in the functional space $A(S_{-\infty})$.

Using this theorem we obtain that every function $F \in A(S_{-\infty})$ can be expanded into the series

$$F(\phi) = \sum_{|\alpha|=0}^{\infty} F_\alpha h_\alpha(\phi) \tag{5}$$

where $F_\alpha = \int_{S_{-\infty}} F(\phi)h_\alpha(\phi)\,\mathrm{d}\nu\,(\phi)/2^{|\alpha|}\alpha!$

Let us introduce on the space of test functions (of an infinite-dimensional argument) $A(S_{-\infty})$ the non-Archimedean norm:

$$\|F\|_2 = \max_\alpha |F_\alpha|_p\sqrt{|2^{|\alpha|}\alpha!|_p}.$$

Let us consider the completion of the space $A\,(S_{-\infty})$ with respect to this norm. We denote this completion by the symbol $L_2(S_{-\infty}, \mathrm{d}\nu)$ and we call the elements of this space square integrable functions with respect to the Gaussian distribution ν. It is a Z_κ-Hilbert space of the type $H_{2^{|\alpha|}\alpha!}$. The inner product (4) is extended continuously from the space of test functions $A(S_{-\infty})$ to the space of square integrable functions $L_2(S_{-\infty}, \mathrm{d}\nu)$ and, in particular, for every function $F \in L_2(S_{-\infty}, \mathrm{d}\nu)$ the integral

$$(F, F) = \int_{S-\infty} |F(\phi)|^2 \, \mathrm{d}\nu\,(\phi) \in Q_p.$$

is well defined.

There is no problem in seeing that

$$L_2(S_{-\infty}, \mathrm{d}\nu) = \left\{ F(\phi) = \sum_{|\alpha|=0}^{\infty} F_\alpha h_\alpha(\phi) : \text{ the series } (F, F) \right.$$
$$\left. = \sum_\alpha |F_\alpha|^2 2^{|\alpha|} \alpha! \text{ converge in } Q_p \right\}.$$

6. The representation of the canonical commutation relation in the space $L_2(S_{-\infty}, \mathrm{d}\nu)$

We use the functional space $L_2(S_{-\infty}, \mathrm{d}\nu)$ as the state space for a quantum system with an infinite number of p-adic degrees of freedom. As usual (see, for example, [15, 16]) we define the field operators

$$\phi_f F(\phi) = (f, \phi) F(\phi) \qquad f \in S_\infty. \tag{6}$$

Then we can define the operators of the Gaussian momentum:

$$\pi_g F(\phi) = (h/\sqrt{\kappa}) D_g F(\phi) \qquad g \in S_{-\infty} \tag{7}$$

where $D_g F(\phi)$ is the derivative of $F(\phi)$ in the direction g and $h \in Q_p$ (h can, in particular, be rational) is a parameter of quantization. Then $[\phi_f, \pi_g] = -h/\sqrt{\kappa}$ as in the real case. We can define the number operator N in the Gaussian representation:

$$N = \tfrac{1}{2} \sum_{j=1}^{\infty} (\pi_i^2 - 2\phi_i \pi_i)$$

where $\pi_i = \pi_{e_i}$, $\phi_i = \phi_{e_i}$ and $\{e_i\}$ is the standard l_2-basis. Then $N h_\alpha(\phi) = |\alpha| h_\alpha(\phi)$ and we can consider $h_\alpha(\phi)$ as pure states with α_1 particles in state $1, \alpha_2$ particles in state $2, \ldots$.

The interpretation of the quantum states with the aid of the number operator N is very useful to build a bridge between p-adic and real quantum fields formalisms. In both formalisms the states $h_\alpha(\phi)$ correspond to the presence of $|\alpha| = \alpha_1 + \alpha_2 + \cdots$ particles.

The superposition principle is used in ordinary quantum field theory to introduce quantum states (5):

$$\sum_{|\alpha|=0}^{\infty} |F_\alpha|^2 \alpha! 2^{|\alpha|} < \infty \qquad F_\alpha \in \mathbb{C} \tag{8}$$

which are (probably infinite) linear combinations of $h_\alpha(\phi)$. There is no problem in any physical application in using quantum states (5) with coefficients $F_\alpha = F_{\alpha 1} + \mathrm{i} F_{\alpha 2}$, $F_{\alpha j} \in Q$.

If $p \neq 1 \pmod 4$, then the square root $\mathrm{i} = \sqrt{-1}$ does not exist in Q_p and we can choose $Q_p(\mathrm{i})$ as the quadratic extension of Q_p. In this case we have $z = x + \mathrm{i}y \in Q_p(i)$, $\bar{z} = x - \mathrm{i}y$, $|z|^2 = x^2 + y^2$. Now formula (7) coincides with the ordinary formula for the momentum. Let us consider the subspace $L_2^Q(Q_p, \mathrm{d}\nu)$ of $L_2(Q_p, \mathrm{d}\nu)$ consisting of functions (5) which have the Hermitian coefficients $F_\alpha = F_\alpha^0 + \mathrm{i} F_\alpha^1$, where $F_\alpha^j \in Q$. The finite sums belong to L_2-spaces both in p-adic and ordinary real case. However, the space $L_2^Q(Q_p, \mathrm{d}\nu)$

contains infinite linear combinations of Hermitian polynomials which do not belong to the ordinary L_2-space, hence the series (8) diverges.

Hence *the p-adic quantum field theory gives us the possibility of extending the superposition principle to realize new quantum states.*

For example,

$$F(\phi) = \sum_{\alpha=0}^{\infty} h_\alpha(\phi) \in L_2(Q_p, \mathrm{d}\nu)$$

for every prime number p. We can also realize much more 'terrible' quantum states:

$$F(\phi) = \sum_{\alpha}^{\infty} (\alpha!)^{\alpha!} h_\alpha(\phi) \in L_2(Q_p, \mathrm{d}\nu).$$

and so on.

Our investigation is only the first step towards a p-adic quantum field theory and there are still many more questions than answers.

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